

Forced oscillations of a body attached to a viscoelastic rod of fractional derivative type

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Abstract

We study forced oscillations of a rod with a body attached to its free end so that the motion of a system is described by two sets of equations, one of integer and the other of the fractional order. To the constitutive equation we associate a single function of complex variable that plays a key role in finding the solution of the system and in determining its properties. This function could be defined for a linear viscoelastic bodies of integer/fractional derivative type.

Keywords: fractional derivative, distributed-order fractional derivative, fractional viscoelastic material, forced oscillations of a rod, forced oscillations of a body

1 Introduction

In this paper we continue our recent work on the dynamics of viscoelastic rods described through the fractional derivative type equations, presented in [4, 5, 6, 7, 8]. A problem that we shall be dealing with in the present paper is forced oscillations of a body attached to a viscoelastic rod with comparable masses. A rod-body system is shown in Figure 1.

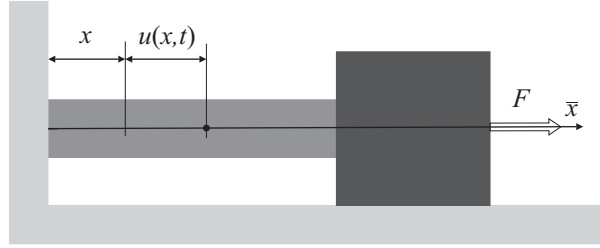


Figure 1: System rod-body.

Our new approach in the investigation of the dynamics of linear viscoelastic rods of fractional type, proposed in [8], is based on the properties of a specially defined function of complex

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variable M (see (18)). Function M is associated with the Laplace transform of the constitutive equation for the material of a rod. It is defined similarly as complex moduli (a quantity obtained after application of Fourier transform to constitutive equation). By considering two cases of constitutive equations, we show in this paper (that is a continuation of [8]) that the proposed approach can be adventitiously used in dynamics of viscoelastic rods.

We find an explicit form of the solution as a convolution of a forcing function and a solution kernel. Moreover, we present numerical examples, corresponding to two common cases of constitutive equations. We analyze forced oscillations of a system consisting of a viscoelastic rod of fractional-order type and a body attached to its end. Thus, the cases of the dynamics of an elastic rod and of a light rod (mass of a rod is negligible) are special cases of our analysis. We refer to [15] for the analysis of oscillations of an elastic rod with the mass attached to its end.

In [4], we analyzed forced oscillations of a body attached to a viscoelastic rod, described by a fractional distributed-order model. We assumed that the mass of a rod is negligible compared to the mass of a body. Similarly as in [4], we analyzed in [6, 7] the wave propagation in a viscoelastic solid-like rod of finite length with one of its ends fixed to a rigid wall. We considered two cases (i.e. two types of boundary conditions): the case when there is a prescribed displacement and the case when there is prescribed stress on rod's free end. Similar problem of wave propagation was analyzed in [5] for a rod made of viscoelastic fluid-like material. The present paper is closely related to [8] where we analyze a more general form of a constitutive equation. Actually, in [8] we gave a theoretical background which we use here and analyze two models which will be described below.

Let m be the mass of a body attached to a rod. The length of the rod in undeformed state is L and its axis, at the initial time moment as well as during the motion, coincides with the \bar{x} axis, see Figure 1. Let x denote a position of a material point of the rod at the initial time $t_0 = 0$. The position of this point at the time $t > 0$ is $x + u(x, t)$. The equations of motion of the rod-body system are

$$\frac{\partial}{\partial x} \sigma(x, t) = \rho \frac{\partial^2}{\partial t^2} u(x, t), \quad \varepsilon(x, t) = \frac{\partial}{\partial x} u(x, t), \quad x \in [0, L], \quad t > 0, \quad (1)$$

$$\int_0^1 \phi_\sigma(\gamma) {}_0D_t^\gamma \sigma(x, t) d\gamma = E \int_0^1 \phi_\varepsilon(\gamma) {}_0D_t^\gamma \varepsilon(x, t) d\gamma, \quad x \in [0, L], \quad t > 0, \quad (2)$$

$$u(x, 0) = 0, \quad \frac{\partial}{\partial t} u(x, 0) = 0, \quad \sigma(x, 0) = 0, \quad \varepsilon(x, 0) = 0, \quad x \in [0, L], \quad (3)$$

$$u(0, t) = 0, \quad -A\sigma(L, t) + F(t) = m \frac{\partial^2}{\partial t^2} u(L, t), \quad t > 0. \quad (4)$$

We use symbols σ , u and ε , in the equation of motion (1)₁ and in the strain (1)₂, to denote stress, displacement and strain, respectively, depending on the initial position x at time t , while ρ denotes the density of a material. Constitutive equation (2) corresponds to the distributed-order fractional derivative model of a viscoelastic body, E is a generalized Young modulus (a positive constant having the dimension of a stress), ϕ_σ and ϕ_ε are given constitutive functions or distributions, ${}_0D_t^\gamma$ is the left Riemann-Liouville fractional derivative operator of order $\gamma \in (0, 1)$ (see [21])

$${}_0D_t^\gamma y(t) := \frac{d}{dt} \left(\frac{t^{-\gamma}}{\Gamma(1-\gamma)} * y(t) \right), \quad t > 0,$$

where Γ is the Euler gamma function and $*$ is the convolution. Recall, if $f, g \in L_{loc}^1(\mathbb{R})$, $\text{supp } f, g \subset [0, \infty)$, then $(f * g)(t) := \int_0^t f(\tau) g(t - \tau) d\tau$, $t \in \mathbb{R}$. We refer to [10, 16, 21] for the basic definitions and assertions of fractional calculus. Initial conditions in (3) specify that the rod-body system is unstressed and in the state of rest at the initial time instant. Boundary

condition $(4)_1$ means that one end of the rod is fixed. The other boundary condition $(4)_2$ is the equation of translatory motion along the \bar{x} axis of the body attached to the free end of the rod. In (4), A stands for the cross-section area of the rod and F stands for the known external force acting on the body.

Regarding the constitutive equation (2) we consider the following two cases.

I Fractional Zener model of a viscoelastic body

$$(1 + a {}_0D_t^\alpha) \sigma(x, t) = E (1 + b {}_0D_t^\alpha) \varepsilon(x, t). \quad (5)$$

It is obtained from (2) by choosing

$$\phi_\sigma(\gamma) := \delta(\gamma) + a \delta(\gamma - \alpha), \quad \phi_\varepsilon(\gamma) := \delta(\gamma) + b \delta(\gamma - \alpha), \quad \alpha \in (0, 1), \quad 0 < a \leq b, \quad (6)$$

where δ denotes the Dirac delta distribution.

II Distributed-order model of a viscoelastic body

$$\int_0^1 a^\gamma {}_0D_t^\gamma \sigma(x, t) d\gamma = E \int_0^1 b^\gamma {}_0D_t^\gamma \varepsilon(x, t) d\gamma. \quad (7)$$

It is obtained from (2) by choosing

$$\phi_\sigma(\gamma) := a^\gamma, \quad \phi_\varepsilon(\gamma) := b^\gamma, \quad \gamma \in (0, 1), \quad 0 < a \leq b. \quad (8)$$

We note that (8) is the simplest form of ϕ_σ and ϕ_ε providing dimensional homogeneity.

Equation (5) is often used in modeling viscoelastic bodies. It is used in [11] for the study of the wave propagation in an unbounded domain. Equation (7) is used in [2, 13] as well as in [6, 7], where the wave motion, stress relaxation and creep, are studied on a bounded domain. We also mention that the wave motion in a body, described by a more general model than (5) is studied in [12]. We refer to [14, 17, 20] for the detailed account of applications of fractional calculus in viscoelasticity. Problems similar to (1) - (4) were also treated in [18, 19] with the constitutive equations related to the distributed-order model (2) in the special cases.

We treated in [7] the creep test of a material described by the constitutive equation (7) and concluded (according to numerical examples) that the material is solid-like, while in [5] the constitutive function was given by

$$(1 + a {}_0D_t^\alpha) \sigma(x, t) = E \left(b_0 {}_0D_t^{\beta_0} + b_1 {}_0D_t^{\beta_1} + b_2 {}_0D_t^{\beta_2} \right) \varepsilon(x, t), \quad (9)$$

where a, b_0, b_1, b_2 are positive constants, $0 < \alpha < \beta_0 < \beta_1 < \beta_2 \leq 1$, and the conclusion was (again based upon numerical examples) that the material is fluid-like. The constitutive equation (9) is proposed in [22]. In this work we treat numerically the creep test for solid-like materials described by (5).

Remark 1 *Constitutive functions (or distributions) ϕ_σ and ϕ_ε in (2) have to satisfy the restrictions following from the Second Law of Thermodynamics, see [1, 3, 5]. We refer to [5] for a systematic review of restrictions if ϕ_σ and ϕ_ε are given in the form of sums of the Dirac δ distributions*

$$\phi_\sigma(\gamma) := \sum_{n=0}^N a_n \delta(\gamma - \alpha_n), \quad \phi_\varepsilon(\gamma) := \sum_{m=0}^M b_m \delta(\gamma - \beta_m), \quad \alpha_n, \beta_m \in [0, 1].$$

Remark 2 *Differences between solid and fluid-like materials are observed in the creep test (i.e. when a material is subjected to a sudden, but later constant force on its free end). Namely, solid-like materials creep to a finite displacement, while the fluid-like materials creep to an infinite displacement.*

The paper is organized as follows. In § 2 we write the system (1) - (4) in the dimensionless form and obtain (10) - (13). Then we formally apply the Laplace transform to (10) - (13), define the function M and obtain the solutions to (10) - (13) in the Laplace domain via the forcing term and solution kernel. Section 3 is devoted to the verification that the function M in the cases of the fractional Zener (5) and distributed-order model (7) satisfies assumptions, cited from [8], that imply the existence and uniqueness of the solutions to (10) - (13). Then, we write theorems on existence and uniqueness of solutions, that are proven in [8]. The explicit form of the solution u , given in Theorem 4, is used in § 4 in order to plot the solution. The plots are given and discussed for the fractional Zener model and for two different forcing functions.

2 Formal solutions

We start from the system (1) - (4) and write it in the dimensionless form. Then, by the Laplace transform method, we obtain the displacement u and the stress σ as the convolution of the external force F and solution kernels P and Q , respectively. Determination of P and Q will be given in § 3.

The system (1) - (4) transforms into

$$\frac{\partial}{\partial x} \sigma(x, t) = \kappa^2 \frac{\partial^2}{\partial t^2} u(x, t), \quad \varepsilon(x, t) = \frac{\partial}{\partial x} u(x, t), \quad x \in [0, 1], \quad t > 0, \quad (10)$$

$$\int_0^1 \phi_\sigma(\gamma) {}_0D_t^\gamma \sigma(x, t) d\gamma = \int_0^1 \phi_\varepsilon(\gamma) {}_0D_t^\gamma \varepsilon(x, t) d\gamma, \quad x \in [0, 1], \quad t > 0, \quad (11)$$

$$u(x, 0) = 0, \quad \frac{\partial}{\partial t} u(x, 0) = 0, \quad \sigma(x, 0) = 0, \quad \varepsilon(x, 0) = 0, \quad x \in [0, 1], \quad (12)$$

$$u(0, t) = 0, \quad -\sigma(1, t) + F(t) = \frac{\partial^2}{\partial t^2} u(1, t), \quad t > 0. \quad (13)$$

This is done by introducing the square root of the ratio between the masses of a rod and a body

$$\kappa = \sqrt{\frac{\rho AL}{m}}$$

and dimensionless quantities

$$\bar{x} = \frac{x}{L}, \quad \bar{t} = \frac{t}{\sqrt{\frac{mL}{AE}}}, \quad \bar{u} = \frac{u}{L}, \quad \bar{\sigma} = \frac{\sigma}{E}, \quad \bar{\phi}_\sigma = \frac{\phi_\sigma}{\left(\sqrt{\frac{mL}{AE}}\right)^\gamma}, \quad \bar{\phi}_\varepsilon = \frac{\phi_\varepsilon}{\left(\sqrt{\frac{mL}{AE}}\right)^\gamma}, \quad \bar{F} = \frac{F}{AE}.$$

In writing (10) - (13) we omitted bar over dimensionless quantities. Note that the choice of dimensionless quantities implies that the case of a rod without the attached mass ($m = 0$) cannot be studied as a special case of equations (10) - (13).

In order to solve the system (10), (11) subjected to the initial (12) and boundary data (13), we use the Laplace transform method. Recall that the Laplace transform of $f \in L_{loc}^1(\mathbb{R})$, $f \equiv 0$ in $(-\infty, 0]$ and $|f(t)| \leq ce^{kt}$, $t > 0$, for some $k > 0$, is defined by

$$\tilde{f}(s) = \mathcal{L}[f(t)](s) := \int_0^\infty f(t) e^{-st} dt, \quad \operatorname{Re} s > k$$

and analytically continued into the appropriate domain D . Moreover, we consider our problems within the space of tempered generalized functions supported by $[0, \infty)$, denoted by \mathcal{S}'_+ . The Laplace transform within this space is an extension of the classical one, given above. Namely, any $g \in \mathcal{S}'_+$ of the form $g := {}_0D_t^\gamma f$, $\gamma \in (0, 1)$, where f is as above (polynomially bounded) satisfies $\mathcal{L}[g(t)](s) = s^\gamma \tilde{f}(s)$, $\text{Re } s > 0$. We refer to [23] for the properties of elements of \mathcal{S}'_+ and their Laplace transforms.

Applying formally the Laplace transform to (10) - (13) we obtain

$$\frac{\partial}{\partial x} \tilde{\sigma}(x, s) = \kappa^2 s^2 \tilde{u}(x, s), \quad \tilde{\varepsilon}(x, s) = \frac{\partial}{\partial x} \tilde{u}(x, s), \quad x \in [0, 1], \quad s \in D, \quad (14)$$

$$\tilde{\sigma}(x, s) \int_0^1 \phi_\sigma(\gamma) s^\gamma d\gamma = \tilde{\varepsilon}(x, s) \int_0^1 \phi_\varepsilon(\gamma) s^\gamma d\gamma, \quad x \in [0, 1], \quad s \in D, \quad (15)$$

$$\tilde{u}(0, s) = 0, \quad \tilde{\sigma}(1, s) + s^2 \tilde{u}(1, s) = \tilde{F}(s), \quad s \in D. \quad (16)$$

By (15) we have

$$\tilde{\sigma}(x, s) = \frac{1}{M^2(s)} \tilde{\varepsilon}(x, s), \quad s \in D, \quad (17)$$

where we introduced

$$M(s) := \sqrt{\frac{\int_0^1 \phi_\sigma(\gamma) s^\gamma d\gamma}{\int_0^1 \phi_\varepsilon(\gamma) s^\gamma d\gamma}}, \quad s \in D. \quad (18)$$

Thus, using (6) and (8), in the cases of constitutive equations (5) and (7) we obtain

$$M(s) = \sqrt{\frac{1 + as^\alpha}{1 + bs^\alpha}}, \quad s \in \mathbb{C} \setminus (-\infty, 0], \quad 0 < a \leq b, \quad \alpha \in (0, 1), \quad (19)$$

$$M(s) = \sqrt{\frac{\ln(bs) as - 1}{\ln(as) bs - 1}}, \quad s \in \mathbb{C} \setminus (-\infty, 0], \quad 0 < a \leq b. \quad (20)$$

So, in the sequel $D = \mathbb{C} \setminus (-\infty, 0]$.

In order to obtain the displacement u , we use (14), (17) and obtain

$$\frac{\partial^2}{\partial x^2} \tilde{u}(x, s) - \kappa^2 s^2 M^2(s) \tilde{u}(x, s) = 0, \quad x \in [0, 1], \quad s \in \mathbb{C} \setminus (-\infty, 0], \quad (21)$$

The solution of (21) is

$$\tilde{u}(x, s) = C_1(s) e^{\kappa x s M(s)} + C_2(s) e^{-\kappa x s M(s)}, \quad x \in [0, 1], \quad s \in \mathbb{C} \setminus (-\infty, 0];$$

C_1 and C_2 are arbitrary functions which are determined from (16)₁ as $2C = C_1 = -C_2$. Therefore,

$$\tilde{u}(x, s) = C(s) \sinh(\kappa x s M(s)), \quad x \in [0, 1], \quad s \in \mathbb{C} \setminus (-\infty, 0]. \quad (22)$$

By (14)₂, (17) and (22) we have

$$\tilde{\sigma}(x, s) = C(s) \frac{\kappa s}{M(s)} \cosh(\kappa x s M(s)), \quad x \in [0, 1], \quad s \in \mathbb{C} \setminus (-\infty, 0]. \quad (23)$$

Using (22) and (23) at $x = 1$, as well as (16)₂ we obtain

$$C(s) = \frac{M(s) \tilde{F}(s)}{s(sM(s) \sinh(\kappa s M(s)) + \kappa \cosh(\kappa s M(s)))}, \quad s \in \mathbb{C} \setminus (-\infty, 0].$$

Therefore, the Laplace transforms of the displacement and stress from (22) and (23) are

$$\tilde{u}(x, s) = \tilde{F}(s) \tilde{P}(x, s), \quad \tilde{\sigma}(x, s) = \tilde{F}(s) \tilde{Q}(x, s), \quad x \in [0, 1], \quad s \in \mathbb{C} \setminus (-\infty, 0]. \quad (24)$$

where

$$\tilde{P}(x, s) = \frac{1}{s} \frac{M(s) \sinh(\kappa x s M(s))}{s M(s) \sinh(\kappa s M(s)) + \kappa \cosh(\kappa s M(s))}, \quad x \in [0, 1], \quad s \in \mathbb{C} \setminus (-\infty, 0], \quad (25)$$

$$\tilde{Q}(x, s) = \frac{\kappa \cosh(\kappa x s M(s))}{s M(s) \sinh(\kappa s M(s)) + \kappa \cosh(\kappa s M(s))}, \quad x \in [0, 1], \quad s \in \mathbb{C} \setminus (-\infty, 0]. \quad (26)$$

Applying the inverse Laplace transform to (24) we obtain the displacement and stress as

$$u(x, t) = F(t) * P(x, t), \quad \sigma(x, t) = F(t) * Q(x, t), \quad x \in [0, 1], \quad t > 0. \quad (27)$$

The validity of these formal expressions will be proved in the sequel.

3 Explicit form of solutions

In order to obtain the displacement u and stress σ by (27), we have to obtain functions P and Q , i.e., to invert the Laplace transform in (25) and (26). First, we examine the behavior of the function M , given by (18), in the limiting cases as $|s| \rightarrow \infty$ and $|s| \rightarrow 0$ (by this we mean only those s that belong to $\mathbb{C} \setminus (-\infty, 0]$) in the special cases when M takes any of the forms given by (19) and (20).

If M is given by (19) or (20) we have

$$|M(s)| \approx \sqrt{\frac{a}{b}}, \quad \text{as } |s| \rightarrow \infty, \quad \text{and } |M(s)| \approx 1, \quad \text{as } |s| \rightarrow 0. \quad (28)$$

We use results from [8] in order to obtain the displacement u and the stress σ . In order to do so, we recall assumptions on M that have to be satisfied.

We shall analyze a function of complex variable

$$f(s) := s M(s) \sinh(\kappa s M(s)) + \kappa \cosh(\kappa s M(s)), \quad s \in \mathbb{C}. \quad (29)$$

Let M be of the form $M(s) = r(s) + i h(s)$, as $|s| \rightarrow \infty$. Then

(A1)

$$\lim_{|s| \rightarrow \infty} r(s) = c_\infty > 0, \quad \lim_{|s| \rightarrow \infty} h(s) = 0, \quad \lim_{|s| \rightarrow 0} M(s) = c_0, \\ \text{for some constants } c_\infty, c_0 > 0.$$

Let $s_n = \xi_n + i \zeta_n$, $n \in \mathbb{N}$, satisfy the equation

$$f(s) = 0, \quad s \in V, \quad (30)$$

where f is given by (29). Then:

(A2) There exists $n_0 > 0$, such that for $n > n_0$

$$\text{Im } s_n \in \mathbb{R}_+ \Rightarrow h(s_n) \leq 0, \quad \text{Im } s_n \in \mathbb{R}_- \Rightarrow h(s_n) \geq 0, \\ \text{where } h := \text{Im } M.$$

(A3) There exist $s_0 > 0$ and $c > 0$ such that

$$\left| \frac{d}{ds}(sM(s)) \right| \geq c, \quad |s| > s_0.$$

(A4) For every $\gamma > 0$ there exists $\theta > 0$ and s_0 such that

$$|(s + \Delta s)M(s + \Delta s) - sM(s)| \leq \gamma, \quad \text{if } |\Delta s| < \theta \text{ and } |s| > s_0.$$

In order to write $M = r + ih$, as required above, we start from $M^2 = u + iv$ and obtain the system

$$r^2 - h^2 = u, \quad 2rh = v.$$

Solutions of the previous system belonging to the set of real numbers are

$$r = \pm \frac{\sqrt{2}}{2} \sqrt{\sqrt{u^2 + v^2} + u}, \quad (31)$$

$$h = \pm \frac{\sqrt{2}}{2} \sqrt{\sqrt{u^2 + v^2} - u}. \quad (32)$$

Assume $\frac{v}{u} \rightarrow 0$, $u > 0$. Then by using the approximation $(1 + x)^a \approx 1 + ax$, as $x \rightarrow 0^+$, from (31) and (32), we have

$$r \approx \pm \sqrt{u}, \quad h \approx \pm \frac{1}{2} \frac{|v|}{\sqrt{u}}. \quad (33)$$

Proposition 3 Functions M given by (19) and (20) satisfy (A1) - (A4).

Proof. Consider M given by (19). If we write $s = re^{i\varphi}$, by (19), we have

$$\begin{aligned} M^2(s) &= \frac{1 + as^\alpha}{1 + bs^\alpha}, \quad a \leq b \\ &= \frac{1 + (a + b)r^\alpha \cos(\alpha\varphi) + abr^{2\alpha}}{1 + 2br^\alpha \cos(\alpha\varphi) + (br^\alpha)^2} - i \frac{(b - a)r^\alpha \sin(\alpha\varphi)}{1 + 2br^\alpha \cos(\alpha\varphi) + (br^\alpha)^2} \\ &\approx \frac{a}{b} - i \frac{b - a}{b^2} \frac{\sin(\alpha\varphi)}{r^\alpha}, \quad \text{as } |s| \rightarrow \infty. \end{aligned}$$

Using (33) we obtain

$$r(s) \approx \pm \sqrt{\frac{a}{b}}, \quad h(s) \approx \pm \frac{1}{2} \sqrt{\frac{b}{a}} \frac{b - a}{b^2} \frac{|\sin(\alpha\varphi)|}{r^\alpha}, \quad \text{as } |s| \rightarrow \infty.$$

Let $\varphi \in (0, \pi)$, then $\sin(\alpha\varphi) > 0$. This implies $\text{Re}(M^2(s)) > 0$ and $\text{Im}(M^2(s)) < 0$. Therefore, we also have $\text{Re}(M(s)) > 0$ and $\text{Im}(M(s)) < 0$. Similar arguments are valid if $\varphi \in (-\pi, 0)$. Hence, we finally have

$$r(s) \approx \sqrt{\frac{a}{b}}, \quad h(s) \approx -\frac{1}{2} \sqrt{\frac{b}{a}} \frac{b - a}{b^2} \frac{\sin(\alpha\varphi)}{r^\alpha}, \quad \text{as } |s| \rightarrow \infty. \quad (34)$$

Next we prove that M in (19) satisfies (A1). By (34) and (19), we have

$$\lim_{|s| \rightarrow \infty} r(s) = \sqrt{\frac{a}{b}}, \quad \lim_{|s| \rightarrow \infty} h(s) = 0, \quad \lim_{|s| \rightarrow 0} |M(s)| = 1.$$

Validity of assumption (A2) follows from (34).

In order to show that M in (19) satisfies (A3), we use (19) and obtain

$$\frac{d}{ds}(sM(s)) = M(s) \left(1 - \frac{\alpha(b-a)s^\alpha}{2(1+as^\alpha)(1+bs^\alpha)} \right).$$

Thus, by (34)

$$\left| \frac{d}{ds}(sM(s)) \right| \approx \sqrt{\frac{a}{b}}, \text{ as } |s| \rightarrow \infty. \quad (35)$$

We have that there exists ξ such that

$$|(s + \Delta s)M(s + \Delta s) - sM(s)| \leq |\Delta s| \left| \left[\frac{d}{ds}(sM(s)) \right]_{s=\xi} \right|.$$

Since $\frac{d}{ds}(sM(s))$, by (35), is bounded as $|s| \rightarrow \infty$ and if $|\Delta s| < \theta$ for some $\theta > 0$, we have that (A4) is satisfied.

Now consider M given by (20). With $s = re^{i\varphi}$ we have

$$\begin{aligned} M^2(s) &= \frac{\ln(bs)}{\ln(as)} \frac{as-1}{bs-1}, \quad a \leq b \\ &= \frac{\ln(ar)\ln(br) + \varphi^2}{\ln^2(ar) + \varphi^2} \frac{abr^2 - (a+b)r\cos\varphi + 1}{(br)^2 - 2br\cos\varphi + 1} - \frac{\varphi \ln \frac{b}{a}}{\ln^2(ar) + \varphi^2} \frac{(b-a)r\sin\varphi}{(br)^2 - 2br\cos\varphi + 1} \\ &\quad - i \left(\frac{\ln(ar)\ln(br) + \varphi^2}{\ln^2(ar) + \varphi^2} \frac{(b-a)r\sin\varphi}{(br)^2 - 2br\cos\varphi + 1} + \frac{\varphi \ln \frac{b}{a}}{\ln^2(ar) + \varphi^2} \frac{abr^2 - (a+b)r\cos\varphi + 1}{(br)^2 - 2br\cos\varphi + 1} \right) \\ &\approx \frac{a}{b} - i \frac{a}{b} \ln \frac{b}{a} \frac{\varphi}{\ln^2(ar)}, \text{ as } |s| \rightarrow \infty. \end{aligned}$$

Using (33) we obtain

$$r(s) = \pm \sqrt{\frac{a}{b}}, \quad h(s) = \pm \frac{1}{2} \sqrt{\frac{a}{b}} \ln \frac{b}{a} \frac{|\varphi|}{\ln^2(ar)}.$$

Let $\varphi \in (0, \pi)$, then $\sin(\alpha\varphi) > 0$. This implies $\operatorname{Re}(M^2(s)) > 0$ and $\operatorname{Im}(M^2(s)) < 0$. Therefore, we also have $\operatorname{Re}(M(s)) > 0$ and $\operatorname{Im}(M(s)) < 0$. Similar arguments are valid if $\varphi \in (-\pi, 0)$. Hence, we finally have

$$r(s) = \sqrt{\frac{a}{b}}, \quad h(s) = -\frac{1}{2} \sqrt{\frac{a}{b}} \ln \frac{b}{a} \frac{\varphi}{\ln^2(ar)}. \quad (36)$$

Next we prove that (20) satisfies (A1). By (36) and (20), we have

$$\lim_{|s| \rightarrow \infty} r(s) = \sqrt{\frac{a}{b}}, \quad \lim_{|s| \rightarrow \infty} h(s) = 0, \quad \lim_{|s| \rightarrow 0} |M(s)| = 1.$$

Validity of assumption (A2) follows from (36).

In order to show M in (20) satisfies (A3), we use (20) and obtain

$$\frac{d}{ds}(sM(s)) = M(s) \left(1 - \frac{\ln \frac{b}{a}}{2 \ln(as) \ln(bs)} + \frac{(b-a)s}{2(as-1)(bs-1)} \right).$$

Thus, by (36)

$$\left| \frac{d}{ds} (sM(s)) \right| \approx \sqrt{\frac{a}{b}}, \text{ as } |s| \rightarrow \infty. \quad (37)$$

Using the same arguments as above, we have that (A4) is satisfied, since $sM(s)$, $s \in V$, by (37), has bounded first derivative. ■

The existence and the uniqueness of u and σ , as solutions to system (10) - (13) is guaranteed by the fact that M in all two cases satisfies (A1) - (A4). Recall, f is given by (29) and s_n , $n \in \mathbb{N}$, are solutions of (30). The multiplicity of zeros s_n is one for n large enough.

Theorem 4 ([8]) *Let $F \in \mathcal{S}'_+$ and suppose that M satisfies assumptions (A1) - (A4). Then the unique solution u to (10) - (13) is given by*

$$u(x, t) = F(t) * P(x, t), \quad x \in [0, 1], \quad t > 0,$$

where

$$\begin{aligned} P(x, t) &= \frac{1}{\pi} \int_0^\infty \operatorname{Im} \left(\frac{M(qe^{-i\pi}) \sinh(\kappa x q M(qe^{-i\pi}))}{q M(qe^{-i\pi}) \sinh(\kappa q M(qe^{-i\pi})) + \kappa \cosh(\kappa q M(qe^{-i\pi}))} \right) \frac{e^{-qt}}{q} dq \\ &\quad + 2 \sum_{n=1}^\infty \operatorname{Re} \left(\operatorname{Res} \left(\tilde{P}(x, s) e^{st}, s_n \right) \right), \quad x \in [0, 1], \quad t > 0, \\ P(x, t) &= 0, \quad x \in [0, 1], \quad t < 0. \end{aligned}$$

The residues are given by

$$\operatorname{Res} \left(\tilde{P}(x, s) e^{st}, s_n \right) = \left[\frac{1}{s} \frac{M(s) \sinh(\kappa x s M(s))}{\frac{d}{ds} f(s)} e^{st} \right]_{s=s_n}, \quad x \in [0, 1], \quad t > 0,$$

Then $P \in C([0, 1] \times [0, \infty))$ and $u \in C([0, 1], \mathcal{S}'_+)$. In particular, if $F \in L^1_{loc}([0, \infty))$, then u is continuous on $[0, 1] \times [0, \infty)$.

The following theorem is related to stress σ . We formulate this theorem with $F = H$, where H denotes the Heaviside function, while the more general cases of F are discussed in Remark 6, below.

Theorem 5 ([8]) *Let $F = H$ and suppose that M satisfies assumptions (A1) - (A4). Then the unique solution σ_H to (10) - (13), is given by*

$$\begin{aligned} \sigma_H(x, t) &= H(t) + \frac{\kappa}{\pi} \int_0^\infty \operatorname{Im} \left(\frac{\cosh(\kappa x q M(qe^{i\pi}))}{q M(qe^{i\pi}) \sinh(\kappa q M(qe^{i\pi})) + \kappa \cosh(\kappa q M(qe^{i\pi}))} \right) \frac{e^{-qt}}{q} dq \\ &\quad + 2 \sum_{n=1}^\infty \operatorname{Re} \left(\operatorname{Res} \left(\tilde{\sigma}_H(x, s) e^{st}, s_n \right) \right), \quad x \in [0, 1], \quad t > 0, \\ \sigma_H(x, t) &= 0, \quad x \in [0, 1], \quad t < 0. \end{aligned} \quad (38)$$

The residues are given by

$$\operatorname{Res} \left(\tilde{\sigma}_H(x, s) e^{st}, s_n \right) = \left[\frac{\kappa \cosh(\kappa x s M(s))}{s \frac{d}{ds} f(s)} e^{st} \right]_{s=s_n}, \quad x \in [0, 1], \quad t > 0.$$

In particular, σ_H is continuous on $[0, 1] \times [0, \infty)$.

Remark 6 ([8])

1. The assumption $F = H$ in Theorem 5 can be relaxed by requiring that F is locally integrable and

$$\tilde{F}(s) \approx \frac{1}{s^\alpha}, \quad \text{as } |s| \rightarrow \infty,$$

for some $\alpha \in (0, 1)$. This condition ensures the convergence of the series in (38).

2. If $F = \delta$, or even $F(t) = \frac{d^k}{dt^k} \delta(t)$, one uses σ_H , given by (38), in order to obtain σ as the $k+1$ -th distributional derivative:

$$\sigma = \frac{d^{k+1}}{dt^{k+1}} \sigma_H \in C([0, 1], \mathcal{S}'_+).$$

4 Numerical examples

The displacement u as a solution to (10) - (13) is given in Theorem 4. We present various numerical examples for constitutive models: fractional Zener and distributed-order model of a solid-like viscoelastic body that are distinguished by the form of M : (19) and (20), respectively.

4.1 The case $F = \delta$

In order to plot time dependence of the displacement u for the several points of the rod as well as for the body attached to its free end, we chose the fractional Zener model and the force acting on the body to be the Dirac delta distribution, i.e. $F = \delta$. We fix the parameters describing the rod: $a = 0.2$, $b = 0.6$, $\alpha = 0.45$ and also fix the ratio between the masses of rod and body $\varkappa := \kappa^2 = 1$. Plot of u as a function of time t for various points of a rod is shown in Figure 2. It is

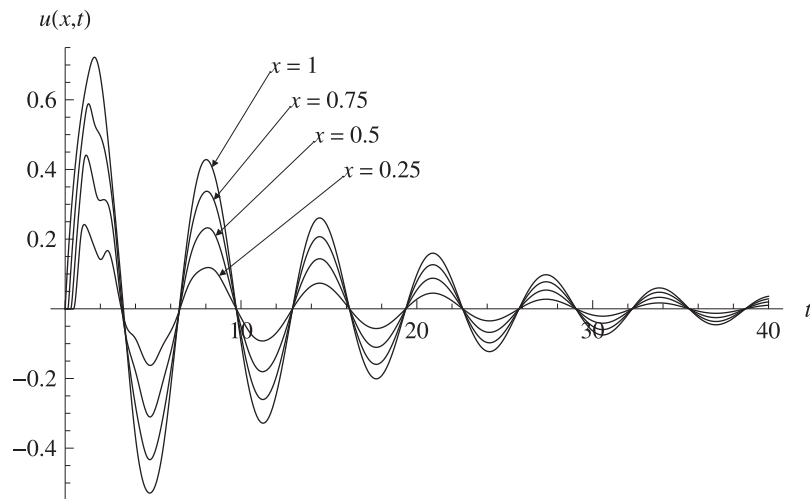


Figure 2: Displacement $u(x, t)$ in the case $F = \delta$ for $\varkappa = 1$ as a function of time t at $x \in \{0.25, 0.5, 0.75, 1\}$ for $t \in (0, 40)$.

evident that the oscillations of the rod and a body are damped, since the material is viscoelastic.

One notices that initially there is a transitional regime of the oscillations. Afterwards, the curves resemble the curves of the damped linear oscillator.

In order to examine the transitional regime more closely, in Figures 3 - 5 we present the plots of u for smaller values of time, but for different values of $\varkappa \in \{0.5, 1, 2\}$. We notice that the

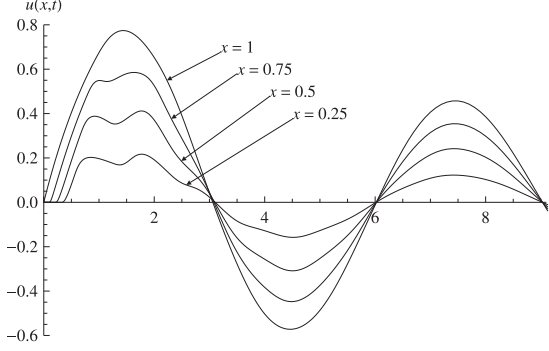


Figure 3: Displacement $u(x, t)$ in the case $F = \delta$ for $\varkappa = 0.5$ as a function of time t at $x \in \{0.25, 0.5, 0.75, 1\}$ for $t \in (0, 9)$.

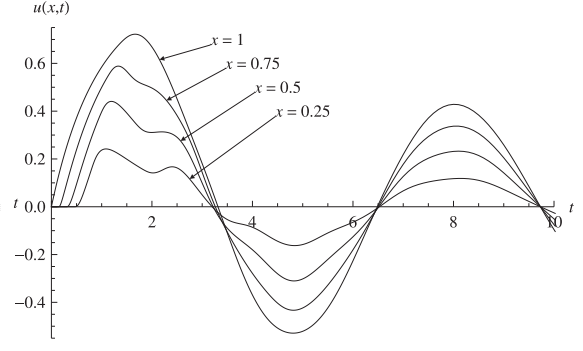


Figure 4: Displacement $u(x, t)$ in the case $F = \delta$ for $\varkappa = 1$ as a function of time t at $x \in \{0.25, 0.5, 0.75, 1\}$ for $t \in (0, 10)$.

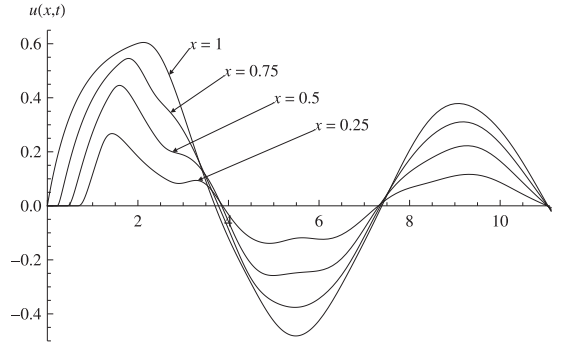


Figure 5: Displacement $u(x, t)$ in the case $F = \delta$ for $\varkappa = 2$ as a function of time t at $x \in \{0.25, 0.5, 0.75, 1\}$ for $t \in (0, 11)$.

shape of the curves depends on the ratio between the masses \varkappa , while later, the shape resembles to the shape of curves for damped oscillations. It could be noticed that regardless of the value of \varkappa there is a delay in starting oscillation for the points that are further away from the free end of a rod. This is due to the finite speed of wave propagation. Namely, the body ($x = 1$), which is subject to the action of the force, starts oscillating at $t = 0$, while the delay in the starting time-instant of the oscillation is greater as the point is further from the point where the force acts. Moreover, we see that different points of the rod do not come to their initial position, for the first time, at the same time-instant (see Figure 5). This, as well as the initial delay depend on the mass ratio \varkappa . For the influence of \varkappa on the initial delay compare Figures 3 - 5. Later on, again depending on \varkappa , the motion of the points become synchronized.

Figures 6 and 7 present the plots of displacement u for fixed point of the rod $x = 0.5$ if the ratio between masses \varkappa varies. One sees from Figure 6 that the larger the mass of a rod is

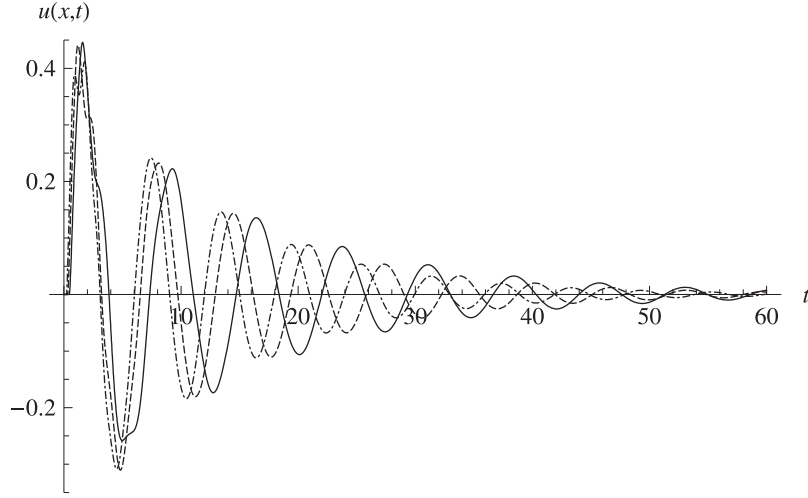


Figure 6: Displacement $u(x, t)$ in the case $F = \delta$ at $x = 0.5$ as a function of time $t \in (0, 40)$ for $\varkappa = 0.5$ - dot-dashed line, $\varkappa = 1$ - dashed line and $\varkappa = 2$ - solid line.

(then the value of \varkappa is greater) the quasi-period (time between two consecutive passage of a fixed point through its initial position) of the oscillations is greater. That is due to the increased rod's inertia. Also, for larger times there is no significant influence of \varkappa on the heights and widths of the peaks which indicates that the damping effects are only due to the parameters a , b and α figuring in the constitutive equation. Figure 7 shows that the shape of the curve in transitional regime strongly depends on \varkappa . Moreover, the delay in the oscillations increases as the \varkappa increases which indicates that the speed of the wave propagation depends on the mass ratio.

4.2 The case $F = H$

The aim of this section is the qualitative analysis of the behavior of a displacement u when there is a force, given in the form of the Heaviside function, i.e., $F = H$, acting at the attached body. Thus, our results correspond to a creep experiment. The rod is modelled by the fractional Zener model, i.e., the function M is given by (19). The parameters describing the rod are: $a = 0.1$, $b = 0.9$, $\alpha = 0.8$. We present plots of u for the ratio between the masses of rod and body $\varkappa = 1$.

Figure 8 present the long-time behavior of displacement u . One notices that the rod creeps to a finite value of displacement so that $\lim_{t \rightarrow \infty} u(x, t) = x$, $x \in [0, 1]$. Figure 9 present the short-time behavior of u . We see that there is a delay in starting time-instant of a point of a rod.

Figures 10 and 11 present the plots of time evolution of displacement u of a rod described by the Zener model for fixed point of the rod $x = 0.5$ if the ratio between masses \varkappa varies. Here, the parameters are $a = 0.2$, $b = 0.6$, $\alpha = 0.45$. One sees, Figure 10, that there is no dependence of the finite value of displacement in creep on the value of the mass ratio \varkappa . We see, Figure 11, that for small times there is an influence of \varkappa on the height of the peaks such that its height increases as \varkappa increases. This is due to the inertia, while for larger times the viscoelastic properties of the rod prevail. Similarly as in the previous section the delay in the oscillations starting time-instant increases as \varkappa increases.

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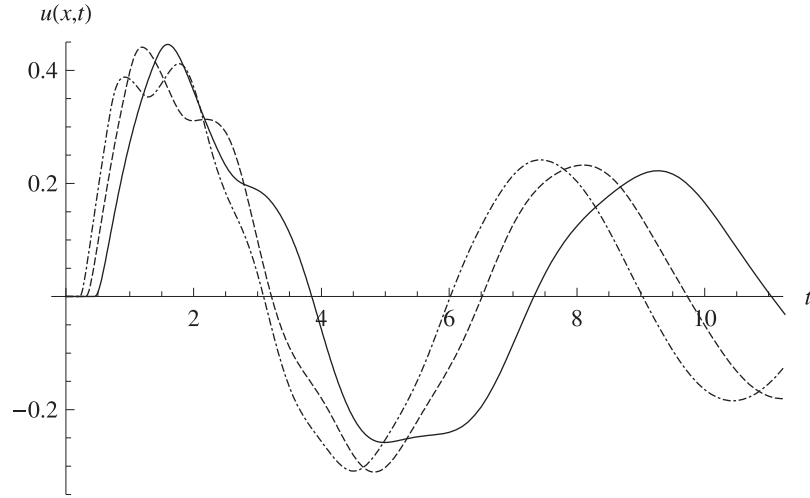


Figure 7: Displacement $u(x, t)$ in the case $F = \delta$ at $x = 0.5$ as a function of time $t \in (0, 11)$ for $\varkappa = 0.5$ - dot-dashed line, $\varkappa = 1$ - dashed line and $\varkappa = 2$ - solid line.

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References

- [1] T.M. Atanackovic, A modified Zener model of a viscoelastic body, *Continuum Mech. Thermodyn.* 14 (2002) 137–148.
- [2] T.M. Atanackovic, A generalized model for the uniaxial isothermal deformation of a viscoelastic body, *Acta Mech.* 159 (2002) 77–86.
- [3] T.M. Atanackovic, On a distributed derivative model of a viscoelastic body, *C. R. Mecanique* 331 (2003) 687–692.
- [4] T.M. Atanackovic, M. Budincevic, S. Pilipovic, On a fractional distributed-order oscillator, *J. Phys. A: Math. Gener.* 38 (2005) 6703–6713.
- [5] T.M. Atanackovic, S. Konjik, Lj. Oparnica, D. Zorica, Thermodynamical restrictions and wave propagation for a class of fractional order viscoelastic rods. *Abstract and Applied Analysis* 2011 (2011) ID975694, 32p.
- [6] T.M. Atanackovic, S. Pilipovic, D. Zorica, Distributed-order fractional wave equation on a finite domain. Stress relaxation in a rod. *Int. J. Eng. Sci.* 49 (2011) 175–190.
- [7] T.M. Atanackovic, S. Pilipovic, D. Zorica, Distributed-order fractional wave equation on a finite domain: creep and forced oscillations of a rod. *Continuum Mech. Thermodyn.* 23 (2011) 305–318.
- [8] T.M. Atanackovic, S. Pilipovic, D. Zorica, On a system of equations arising in viscoelasticity theory of fractional type. Preprint available on arXiv:1205.5343 (2012).

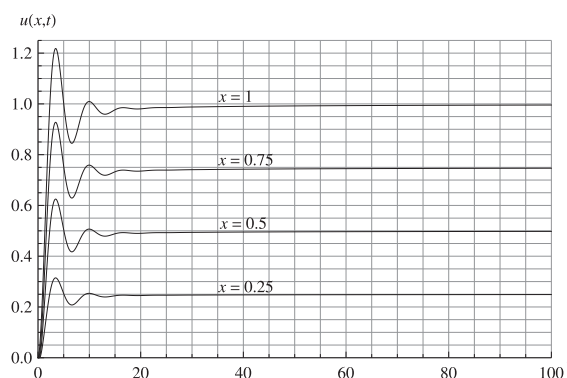


Figure 8: Displacement $u(x, t)$ in the creep experiment for $\varkappa = 1$ as a function of time t at $x \in \{0.25, 0.5, 0.75, 1\}$ for $t \in (0, 100)$.

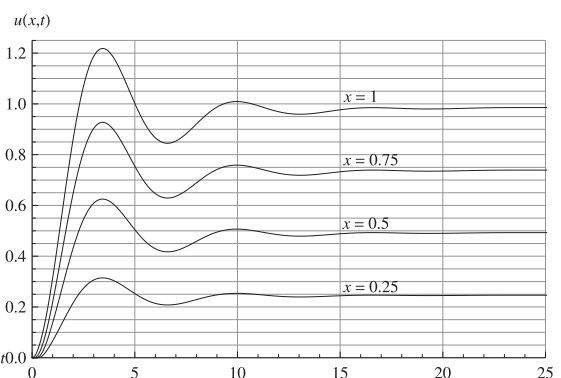


Figure 9: Displacement $u(x, t)$ in the creep experiment for $\varkappa = 1$ as a function of time t at $x \in \{0.25, 0.5, 0.75, 1\}$ for $t \in (0, 25)$.

- [9] G. Doetsch, Handbuch der Laplace-Transformationen I. Birkhauser, Basel, 1950.
- [10] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations. Elsevier B.V, Amsterdam, 2006.
- [11] S. Konjik, Lj. Oparnica, D. Zorica, Waves in fractional Zener type viscoelastic media. J. Math. Anal. Appl. 365 (2010) 259–268.
- [12] S. Konjik, Lj. Oparnica, D. Zorica, Waves in viscoelastic media described by a linear fractional model. Integr. Transf. Spec. F. 22 (2011) 283–291.
- [13] T.T. Hartley, C.F. Lorenzo, Fractional-order system identification based on continuous order-distributions, Signal Process. 83 (2003) 2287–2300.
- [14] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity. Imperial College Press, London, 2010.
- [15] W. Nowacki, Dynamics of elastic systems. Chapman & Hall, London 1963.
- [16] I. Podlubny, Fractional Differential Equations. Academic Press, San Diego, 1999.
- [17] Yu.A. Rossikhin, Reflections on two parallel ways in the progress of fractional calculus in mechanics of solids. Applied Mechanics Reviews, 63 (2010) 010701-1–010701-12.
- [18] Yu.A. Rossikhin, M.V. Shitikova, Analysis of dynamic behavior of viscoelastic rods whose rheological models contain fractional derivatives of two different orders. Zeitschrift für Angewandte Mathematik und Mechanik, 81 (2001) 363–376.
- [19] Yu.A. Rossikhin, M.V. Shitikova, A new method for solving dynamic problems of fractional derivative viscoelasticity. International Journal of Engineering Science, 39 (2001) 149–176.
- [20] Yu.A. Rossikhin, M.V. Shitikova, Application of fractional calculus for dynamic problems of solid mechanics: Novel trends and recent results. Applied Mechanics Reviews, 63 (2010) 010801-1–010801-52.
- [21] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives, Gordon and Breach, Amsterdam, 1993.

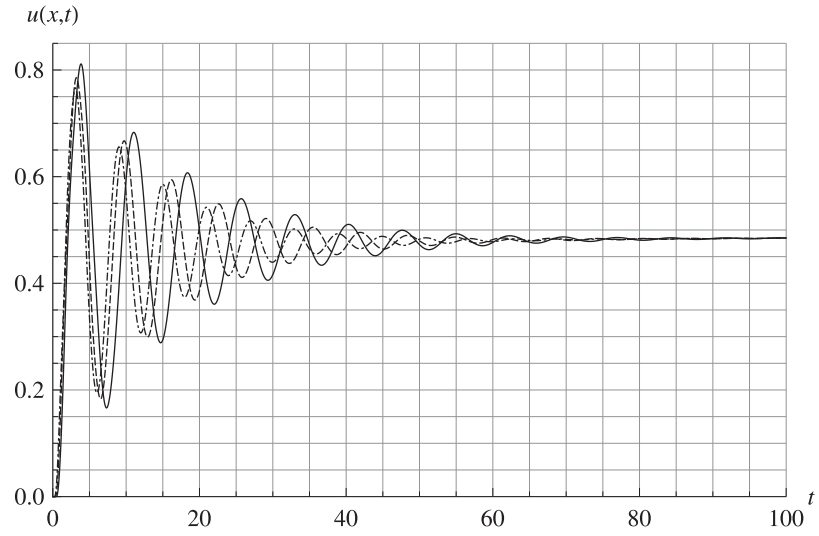


Figure 10: Displacement $u(x, t)$ in the creep experiment at $x = 0.5$ as a function of time $t \in (0, 100)$ for $\varkappa = 0.5$ - dot-dashed line, $\varkappa = 1$ - dashed line and $\varkappa = 2$ - solid line.

[22] H. Schiessel, Chr. Friedrich, A. Blumen, Applications to problems in polymer physics and rheology. In *Applications of fractional calculus in physics* (ed R. Hilfer). World Scientific, Singapore, 2000.

[23] V.S. Vladimirov, Equations of Mathematical Physics, Mir Publishers, Moscow, 1984.

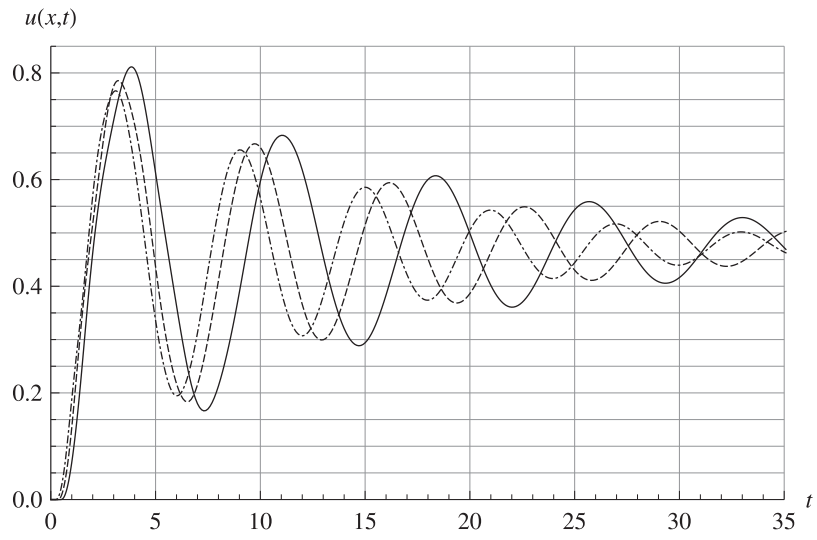


Figure 11: Displacement $u(x, t)$ in the creep experiment at $x = 0.5$ as a function of time $t \in (0, 35)$ for $\varkappa = 0.5$ - dot-dashed line, $\varkappa = 1$ - dashed line and $\varkappa = 2$ - solid line.